COMPLETELY γ_r ENRESDOWED GRAPHS

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Abstract: Let G = (V, E) be a non empty, finite, simple graph. A dominating set of a graph G containing a minimum dominating set of G is called a γ - endowed dominating set of G. If that set is of cardinality k then it is called a k γ – endowed dominating set. k - γ_r enresdowed graph is one in which every restrained dominating set of cardinality k contains a minimum restrained dominating set. Completely γ_r enresdowed graph is a graph in which for every restrained dominating set of cardinality k , $\gamma_r \leq k \leq n - 2$, G is k - γ_r enresdowed . The graphs in which every minimal restrained dominating set is minimum is called well restrained dominated graph . In this paper we found few results for the completely γ_r enresdowed graph and well restrained dominated graphs.

Keywords: Completely γ_r enresdowed graphs, Enresdowed graphs, Well restrained dominated graphs.

Introduction: Let G = (V, E) be a non empty, finite, simple graph. A subset D of V(G) is called a dominating set of G if for every v ∈ V – D, there exists u ∈ D such that u and v are adjacent. The minimum cardinality of the dominating set is called the domination number and it is denoted by γ(G)[2]. The restrained dominating set of a graph is a dominating set in which every vertex in V – D is adjacent to some other vertex in V – D. The minimum cardinality of the restrained dominating set is called the restrained domination number and it is denoted by γ_r (G)[7]. A graph is said to be k - γ_r enresdowed graph if every restrained dominating set of cardinality k contains a minimum restrained dominating set[8]. Completely $\gamma_r$enresdowed graph is one in which for every cardinality k, $\gamma_r \leq k \leq n - 2$ G is k - γ_r enresdowed . A dominating set D of G is called a minimal dominating set if no proper subset of D is a dominating set of G[3]. A restrained dominating set D of G is called a minimal restrained dominating set if no proper subset of D is a restrained dominating set of G.

Definition 1.1: Let k be a positive integer . A simple, finite, non trivial graph G = (V, E) is called a k - γ_r enresdowed graph if every restrained dominating set of G of cardinality k contains a minimum restrained dominating set γ_r of G[9].

Definition 1.2 A graph G is said to be completely γ_r enresdowed if for every k, $\gamma_r \leq k \leq n - 2$, G is k - γ_r enresdowed.

Result 1.3

1. K_n is completely γ_r enresdowed for any k, where $\gamma_r \leq k \leq n - 2$.
2. C_n, $n \geq 6$, is not completely γ_r enresdowed.
3. $\overline{K_n}$ is completely γ_r enresdowed for any $k = n$.

Proposition 1.4

Let C_n be a cycle of order n, for n = 3,4,5, then C_n is completely γ_r enresdowed for any k ,where $\gamma_r \leq k \leq n - 2$.

Proof

Consider the cycle C_n, n = 3,4,5. and $\{v_1, v_2,..., v_i , ... , v_n\}$, $1 \leq i \leq n$, $3 \leq n \leq 5$ be the vertex set of C_n. Then the γ_r set of C_n is $\{v_1, v_2,..., v_j , ... , v_{n-2}\}$, $1 \leq j \leq n - 2$, thus C_n is k - γ_r enresdowed for $k = \gamma_r = n - 2$, and for $k = \gamma_r + 1$,it is not k - γ_r enresdowed . Hence C_n is completely γ_r enresdowed.

Theorem 1.5:

Let $K_{m,n}$, where m = n be a complete bipartite graph. Then $K_{m,n}$ is completely γ_r enresdowed for any k, where $\gamma_r \leq k \leq n - 2$.

Proof: Let $K_{m,n}$,where m = n be a complete bipartite graph. The vertex set V of $K_{m,n}$, is partioned into two subsets $V_1$and V_2, where $V_1 = \{v_1, v_2,..., v_i , ... , v_m\}$, $1 \leq i \leq m$ and $V_2 = \{w_1, w_2,..., w_j , ... , w_n\}$, $1 \leq j \leq n$, such that $|V_1| = m$ and $|V_2| = n$.
Let $D_1= \{v_i, w_j\}$, $1 \leq i \leq m$ and $1 \leq j \leq n$ be the γ_r set of $K_{m,n}$. Thus $K_{m,n}$ is k - γ_r enresdowed for $k = \gamma_r$, without loss of generality consider the set D_2, where $D_2 = D_1 \cup \{v_l\}$ $1 \leq l \leq m - 1$, and $l \neq i$, where D_2 is k - γ_r enresdowed for $k = \gamma_r +1$. Similarly consider the set $D_3 = D_1 \cup\{v_r,w_s\}$, $1 \leq r \leq m - 1$, $r \neq i$ and $1 \leq s \leq n - 1$, $s \neq j$, then D_3 is k - γ_r enresdowed for $k = \gamma_r +2$, also the sets $D_4 = D_1 \cup\{v_r,v_{r+1}\}$, $1 \leq r \leq m - 1$ and $D_5 = D_1 \cup\{w_s,w_{s+1}\}$, $1 \leq s \leq n - 1$ is k - γ_r enresdowed for $k = \gamma_r +2$. Finally consider the set D_t, where $D_t = \{v_i\}\cup\{w_j\}$,$1 \leq i \leq m-1$ and $1 \leq j \leq n - 1$. Thus D_t is k - γ_r enresdowed for $k = n - 2$. Thus $K_{m,n}$ is completely γ_r enresdowed.

Well Restrained Dominated Graphs:

Definition 2.1: A minimal restrained dominating set D on a graph G is a restrained dominating set D which have the property that no proper subset of D is also a restrained dominating set.

Definition 2.2: $\gamma_r(G)$ is the cardinality of the smallest minimal restrained dominating set and $\Gamma_r(G)$ is the cardinality of the largest minimal restrained dominating set.

Definition 2.3: A graph G is said to be well restrained dominated if every minimal restrained dominating set is minimum.

Theorem 2.4

Let G be a cycle C_n , n = 3,4,5, then G is a well restrained dominating graph.

Proof: Given G is a cycle C_n, n = 3,4,5. Let $\{v_1, v_2,..., v_i , ... , v_n\}$, $1 \leq i \leq n$, $3 \leq n \leq 5$ be the vertex set of C_n. Let $D =\{v_1, v_2,..., v_i , ... , v_{n-2}\}$, $1 \leq i \leq n - 2$ be the γ_r set of C_n of cardinality k, where $k = \gamma_r$. Thus the γ_r set D is the minimal restrained dominating set of C_n. Also the set D is minimum for any C_n, n = 3,4,5. Consider the set $D_1$of cardinality k_1, where $k_1= \gamma_r+1 = n-1$, where $D_1=\{v_1, v_2,..., v_i , ... , v_{n-1}\}$, $1 \leq i \leq n - 1$.Then there exist an isolated vertex in V - D_1. Thus D_1 is not a restrained dominating set,further it is not minimal. Similarly consider the set $D_2= \{v_1, v_2,..., v_i , ... , v_n\}$,of cardinality k_2 , where $k_2= n$, where D_2 is not minimal restrained dominating set. Thus for C_n, n = 3,4,5 the γ_r set D is the only minimal restrained dominating set which is also minimum. Hence G is a well restrained dominating graph.

Theorem 2.5: Let G be any cycle graph C_n, $n \geq 6$, then G is not a well restrained dominating graph.

Proof: Given G is a cycle C_n, $n \geq 6$. Let $\{v_1, v_2,...,v_{i-1}, v_i , v_{i+1} ..., v_{n-1}, v_n\}$ be the vertex set of C_n, $n \geq 6$. Let $D =\{v_1, v_4,..., v_j , ... , v_n\}$, j = 1,4,7,...,n be the γ_r set of C_n, $n \geq 6$ of cardinality k, where $k = \gamma_r$ then the γ_r set D is the minimal restrained dominating set of C_n, also the set D is minimum for any C_n, $n \geq 6$.

Now we consider a restrained dominating set D_1 of cardinality k_1, where $k_1 \geq \gamma_r + j$, j is even. Let the vertex set of D_1be,where $D_1=\{v_1, v_4,... , ...,v_{n-3}, v_{n-2},v_{n-1} , v_n\}$, where V - D_1 is $\{v_2, v_3, v_5, v_6,... , ... v_{n-5}, v_{n-4}\}$, and $k_1> k = \gamma_r$. Consider a set D_{11} from D_1 by deleting any vertex v_l,where $D_{11}= D_1 - \{v_l\}$, l = 1,4,7,...,n then the set , V - $D_{11} = \{V - D_1 \}\cup\{v_l\}$, l = 1,4,7,...,n . Since the vertex $v_l \in D_1$ is adjacent to v_{l-1} and v_{l+1} , where v_{l-1} and v_{l+1} belong to the set V - D_{11} , every vertex in V - D_{11} are adjacent to some other vertex in V - D_{11} . Thus there exist a vertex v_r in V - D_{11} which is not dominated by any vertices of D_{11} . Hence D_{11} is not a restrained dominating set.

Without loss of generality consider another set, $D_{12} = D_1 - \{v_s\}$, s = 1,4,7,...,n there exist a vertex v_s in D_1 which is adjacent to v_{s-1} and v_{s+1} , where v_{s-1} and v_{s+1} belong to the set D_{12} , then v_s is isolate vertex in V - D_{12} . Thus the set D_{12} is not restrained dominating set. Thus the set D_1 is minimal restrained dominating set of cardinality $k_1> k = \gamma_r$ where D_1 is not minimum . Hence there exist a minimal restrained dominating set which is not minimum . Thus G is not a well restrained dominating graph.

Results 2.6:
1. For any path P_n , 2≤ n ≤ 6 is well restrained dominating graph.
2. For any path P_n, n ≥ 7 is not well restrained dominating graph.
3. A complete bipartite graph is well restrained dominating graph .

Definition 2.7: Comb graph is a graph obtained by joining a single pendant edge to each vertex of a path.[8]

Theorem 2.9: Let G be a comb graph with a path P_n , $n \geq 2$. Let $\{v_i\}$, $1 \leq i \leq n$, be the set of vertices of P_n and $\{w_i\}$, $1 \leq i \leq n$, be the set of pendant vertices of G. Let D be the γ_r set of G and p = 2n. Then G is completely γ_r enresdowed for any *k*, where $\gamma_r \leq k \leq p-2$ and G is a well restrained dominated graph.
Proof: Given G is a comb graph with a path P_n , $n \geq 2$. Let $\{v_1, v_2,..., v_i , ... , v_n\}$, $1 \leq i \leq n$ be the vertices of the path P_n of G and the set $\{w_1, w_2,..., w_i , ... , w_n\}$, $1 \leq i \leq n$ be the set of pendants of G, where each w_i , $1 \leq i \leq n$ is attached to v_i, $1 \leq i \leq n$. Let D be the γ_r set of G and p = 2n. Choose a set of vertices $\{w_1, w_2,..., w_{i-1}, w_i , w_{i+1}, ... w_{n-1}, w_n\}$, $1 \leq i \leq n$ for the γ_r set D, where the set of vertices of the path $\{v_1, v_2,..., v_{i-1}, v_i , v_{i+1} , ... v_{n-1}, v_n\}$, $1 \leq i \leq n$ are dominated , also every vertex in V – D is adjacent in V – D. Thus the set of vertices $\{w_1, w_2,..., w_{i-1}, w_i , w_{i+1}, ... w_{n-1}, w_n\}$, $1 \leq i \leq n$ forms a restrained dominating set of cardinality $k = \gamma_r$, which contains the minimum restrained dominating set. Hence G is k - γ_r enresdowed for any *k*, where $k = \gamma_r$.

Consider a restrained dominating set D_1, of cardinality k_1, where $k_1 = \gamma_r + 1$. The set D_1 is , $D_1 = D \cup\{v_i\}$, $1 \leq i \leq n$ such that $v_i \notin \{v_2,v_{n-1}\}$. Then the set D_1 forms a restrained dominating set containing the minimum restrained dominating set D of G. Hence G is k_1 - γ_r enresdowed for any k_1, where $k_1 = \gamma_r + 1$.

Consider a restrained dominating set D_2 of cardinality k_2, where $k_2 = \gamma_r + 2$. The set D_2 is , $D_2 = D \cup\{v_r , v_s\}$, $1 \leq r,s \leq n$, then there exist the following cases for the set D_2, where
Case(i) $D_{21} = D \cup\{v_r , v_s\}$, $1 \leq r,s \leq n$, if both the vertices $v_r , v_s \in \{v_2 , v_{n-1}\}$
Case (ii) $D_{22} = D \cup\{v_r , v_s\}$, $1 \leq r,s \leq n$, if either v_r or $v_s \in \{v_2 , v_{n-1}\}$
Case (iii) $D_{23} = D \cup\{v_r , v_s\}$, $1 \leq r,s \leq n$, if $v_s = v_{r+2}$
Case (iv) $D_{24} = D \cup\{v_r , v_s\}$, $1 \leq r,s \leq n$, If v_r , v_s are adjacent vertices.
Case(i)
Consider the set $D_{21} = D \cup\{v_r , v_s\}$, $1 \leq r,s \leq n$, suppose if both v_r , v_s belongs to $\{v_2 , v_{n-1}\}$, then the set V - D_{21} is V - D_{21}= $\{v_1, v_3,..., v_{i-1}, v_i , v_{i+1} , ... v_{n-2}, v_n\}$. Thus the vertices v_1 and v_n of the V - D_{21} set is an isolate vertices of V - D_{21}. Thus the set D_{21} is not a restrained dominating set. Hence G is not k_{21} - γ_r enresdowed for any k_{21}, where $k_{21} = \gamma_r + 2$.
Case (ii)

Consider the set $D_{22} = D \cup\{v_r , v_s\}$, $1 \leq r,s \leq n$, suppose if either v_r or v_s , $1 \leq r,s \leq n$, belong to the set $\{v_2 , v_{n-1}\}$. Without loss of generality, assume that $v_r = v_2$, then there exist an isolate vertex v_1 in V - D_{22}. Thus the set D_{22} is not a restrained dominating set. Hence G is not k_{22} - γ_r enresdowed for any k_{22}, where $k_{22} = \gamma_r + 2$.
Case(iii)

Consider the set $D_{23} = D \cup\{v_r ,v_s\}$, $1 \leq r,s \leq n$, suppose if $v_s = v_{r+2}$, then there exist an isolate vertex v_{r+1} in V - D_{23}. Thus the set D_{23} is not a restrained dominating set. Hence G is not k_{23} - γ_r enresdowed for any k_{23}, where $k_{23} = \gamma_r + 2$.
Case(iv)

Consider the set $D_{24} = D \cup\{v_r ,v_s\}$, $1 \leq r,s \leq n$, suppose if v_r , v_s , $1 \leq r,s \leq n$, are adjacent vertices , such that $v_r , v_s \notin \{v_2 ,v_3\}$ and $v_r ,v_s \notin \{v_{n-2} ,v_{n-1}\}$, suppose if $v_r ,v_s \in \{v_2 ,v_3\}$ and $\{v_{n-2} ,v_{n-1}\}$, then the vertices v_1 and v_n are isolates inV - D_{24}. Thus the set D_{24} is a restrained dominating set which contains the minimum restrained dominating set D. Hence G is k_{24} - γ_r enresdowed for any k_{24}, where $k_{24} = \gamma_r + 2$.Proceeding in the same way, Consider the restrained dominating set D_{n-2} , where D_{n-2} = D ∪ $\{v_1, v_2,..., v_l\}$, $1 \leq l \leq$ n - 2 ,of cardinality k_{n-2}, where k_{n-2} = 2n – 2 . The set D_{n-2} is , D_{n-2} = $\{w_1, w_2,..., w_{i-1}, w_i , w_{i+1}, ... w_{n-1}, w_n, v_1, v_2,..., v_l\}$, $1 \leq l \leq$ n - 2 , then the set V - D_{n-2} has no isolates. Thus D_{n-2} is a restrained dominating set containing the minimum restrained dominating set D of G. Hence G is k_{n-2} - γ_r enresdowed for any k_{n-2}, where k_{n-2}= 2n – 2 = p - 2. Thus G is k - γ_r enresdowed for any k, where $\gamma_r \leq k \leq$ p – 2. Thus G is completely γ_r enresdowed. The set of vertices $\{w_1, w_2,..., w_{i-1}, w_i , w_{i+1}, ... w_{n-1}, w_n\}$, $1 \leq i \leq$ nis the only minimal restrained dominating set of G, which is minimum . Hence G is a well restrained dominated graph.

Conclusion: In this paper, we have demonstrated the hypotheses in view of the completely γ_r enresdowedness property of standard graphs for example , cycle graph, complete graph, bipartite graphs, comb graph and path graph. The hypotheses is demonstrated for cycle and path graph utilizing the well restrained dominated property .

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